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FRW models with a variable G and a vacuum, varying as radiation

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Abstract : With a variable G and a vacuum energy, varying as the radiation content of the universe, we find flat, closed and open FRW models starting either from a non-singular origin with a minimum, non-zero G or from a singularity with a vanishing G. G increases continuously in the radiation dominated era and approaches a constant value as the universe turns matter dominated and the models then approach the standard model

Keywords : FRW models, variable G and Λ, vacuum energy

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1. Introduction

The non-trivial role of vacuum in the early universe generates a Λ-term in Einstein's field equations [1]. By considering the conservation of the energy momentum tensor of matter and vacuum taken together, many authors have invoked the idea of a decaying vacuum energy and hence a varying cosmological 'constant' Λ with cosmic expansion to understand phenomenologically the incredibly small value of Λ [2–4]. In this context, Freese *et al* [3] have discussed a spatially flat FRW model wherein the vacuum energy density ρ_v has been taken to vary as the radiation energy density ρ_r :

$$\rho_v = \frac{x}{(1-x)}\rho_r . \tag{1.1}$$

Here x is a constant lying between 0 and 1 for a genuinely new cosmology and assumes two constant values : one in the radiation dominated (R.D.) and the other in the matter dominated (M.D.) universe.

As the Newtonian constant of gravity G plays the role of a coupling constant between geometry and matter in the Einstein field equations, it appears natural to look at this constant as a function of time in an evolving universe. There are many extensions of Einstein's theory of gravitation in which G is taken to vary with time [5]. Recently, it has been proposed to link the variation of G with that of the cosmological constant Λ leaving the form of the Einstein field equations unchanged and preserving the conservation of the energy momentum tensor of matter content [6]. Though this approach is non-covariant but it is worth studying because it may be a limit of some higher dimensional fully covariant theory

Our aim in the present paper, is to study the different cosmological scenarios which result from the inclusion of a dynamically decaying vacuum energy and a time-dependent gravitational coupling G where the vacuum varies as the radiation content of the universe according to eq. (1.1).

2. The field equations

We consider an isotropic, homogeneous space time given by the Robertson-Walker metric

$$ds^2 = dt^2 + R^2\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right], \; k = \pm 1, 0, \tag{2.1}$$

and a distribution of the matter represented by the perfect fluid energy momentum tensor

$$T_{ij} = (\rho + p) v_i v_j + p g_{ij}. \tag{2.2}$$

The Einstein field equation with time dependent G and Λ, can be written as

$$R_{ij} - \frac{1}{2} R^h_h g_{ij} = -8\pi G(t)\left[T_{ij} - \frac{\Lambda(t)}{8\pi G(t)} g_{ij}\right]. \tag{2.3}$$

Here, $-\frac{\Lambda}{8\pi G} g_{ij}$ is the energy momentum tensor of vacuum with $\frac{\Lambda}{8\pi G} = \rho_v$ being the energy density of vacuum and the isotropic, homogeneous pressure $p_v = -\rho_v$, though other interpretations of the cosmological term also exist in the literature [7].

In the context of eqs. (2.1) and (2.2), the field equation (2.3) obtains

$$\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = \frac{8\pi G(t)}{3}(\rho + \rho_v), \tag{2.4}$$

$$\frac{\ddot{R}}{R} = -\frac{4\pi G(t)}{3}(\rho + 3p - 2\rho_v), \tag{2.5}$$

where the non-vacuum energy density ρ is the sum of rest mass and radiation energy densities :

$$\rho = \rho_m + \rho_r. \tag{2.6}$$

and the pressure of the non-vacuum component is that of radiation only :

$$p = p_r = \frac{1}{3}\rho_r. \tag{2.7}$$

In view of the vanishing divergence of Einstein tensor, eq. (2.3) yields

$$\dot{\rho} + 3(\rho + p)\frac{\dot{R}}{R} + \dot{\rho}_v + (\rho + \rho_v)\frac{\dot{G}}{G} = 0. \tag{2.8}$$

We now assume the law of conservation of energy momentum tensor ($T_{ij}^{;j} = 0$) giving

$$\dot{\rho} + 3(\rho + p)\frac{\dot{R}}{R} = 0, \tag{2.9}$$

which by the use of (2.6) and (2.7), may alternatively be written as

$$\frac{1}{R^4}\frac{d}{dt}\left(\rho_r R^4\right) + \frac{1}{R^3}\frac{d}{dt}\left(\rho_m R^3\right) = 0, \tag{2.10}$$

yielding the solution

$$\left.\begin{aligned} \rho_r &= E_r R^{-4}, \\ \rho_m &= E_m R^{-3}, \end{aligned}\right\} \tag{2.11}$$

where E_r and E_m are some positive constants. By the use of (2.9), eq. (2.8) obtains

$$\frac{\dot{\rho}_v}{\rho + \rho_v} = -\frac{\dot{G}}{G}, \tag{2.12}$$

indicating that $\dot{G}$ increases or decreases according as the vacuum energy density ρ_v respectively decreases or increases. With the help of (1.1) and (2.6), eq. (2.12) may be written as

$$-\frac{\dot{G}}{G} = \frac{\dfrac{x}{(1-x)}\dot{\rho}_r}{\rho_m + \dfrac{\rho_r}{(1-x)}}, \tag{2.13}$$

which, by the use of (2.11), obtains

$$\frac{\dot{G}}{G} = 4xE_r\left\{\frac{\dot{R}}{R[(1-x)E_m R + E_r]}\right\}, \tag{2.14}$$

yielding the solution

$$G = \text{constant.}\left[\frac{R}{E_m(1-x)R + E_r}\right]^{4x} \tag{2.15}$$

We shall now utilize these equations to describe the different phases of evolution of the model which will be done in the following Sections starting with the early phase of pure radiation.

3. The early universe

In the early pure radiation era, eq. (2.15) obtains

$$G = AR^{4x}, \tag{3.1}$$

where A is some positive constant. By the use of (1.1), (2.11) and (3.1), eq. (2.4) in this era, reduces to

$$\dot{R}^2 = \frac{8\pi AE_r}{3(1-x)} R^{(4x-2)} - k, \tag{3.2}$$

which gives

$$\ddot{R} = -\frac{8\pi AE_r}{3}\frac{(1-2x)}{(1-x)} R^{(4x-3)}. \tag{3.3}$$

It is hard to integrate eq. (3.2) for a general x unless $k = 0$. However, we can draw some information about the initial stage of the model even without a solution. We note from eq. (3.3) that with $\frac{1}{2} < x < 1$ or equivalently with $\rho_v > \rho_r$ it is possible to avoid $\ddot{R} \le 0$ and hence the initial singularity. The expansion in this case, would be endowed with a generalized inflation ($\ddot{R} \le 0$). For $0 < x \le \frac{1}{2}$ or equivalently for $\rho_v \le \rho_r$, eq. (3.3) gives $\ddot{R} \le 0$ which together with $\dot{R} > 0$ implies that the curve $R(t)$ must have reached $R(t) = 0$ at some finite time in the past where there is a singularity. We thus see that the field equations admit different initial conditions and thereby give rise to a number of cosmological scenarios, some of which are investigated in the following by considering some special values of x *viz.* $\frac{3}{4}$, $\frac{1}{2}$ and $\frac{1}{4}$ when $k \neq 0$ and for a general x when $k = 0$.

(a) An inflationary scenario starting from a non-singular origin:

We first consider $x = 3/4$. Eq. (3.3) then reduces to $\ddot{R}$ = constant = $\frac{16\pi AEr}{3} > 0$ and a possibility arises of $R(t)$ decreasing in the past to a finite minimum radius (say R_0) rather than zero, where $\dot{R}(t) = 0$. We take this time as $t = 0$. (However, in order that this minimum of R may never become zero demands that k should be greater than zero, *i.e.*, $k = 1$.) Eqs. (2.11) and (3.1)–(3.3) then respectively obtain $E_r = \rho_{r0}R_0^4, A = G_0R_0^{-3}, \rho_{r0}G_0R_0^2 = \frac{3k}{32\pi}$ indicating $k = 1$ and $\ddot{R} = \frac{1}{2R_0}$. The model then starts from rest from a non-singular origin with maximum radiation and vacuum energy densities ρ_{r0} and $\rho_{v0} = 3\rho_{r0}$ and a minimum gravitational coupling $G_0 = \frac{3}{32\pi\rho_{r0}R_0^2}$. The expansion is driven by some general (non-exponential) type of inflation throughout the era. For the above stated values of the constants, eq. (3.2) then reduces to

$$\dot{R}^2 = \frac{R}{R_0} - 1, \tag{3.4}$$

yielding the solution

$$R = R_0\left[\frac{t^2}{4R_0^2} + 1\right]. \tag{3.5}$$

The expression for the Hubble parameter $H\ (= \dot{R}/R)$ in this model, may be obtained in the form $H = \frac{2}{t}[1+\frac{R_0}{R-R_0}]^{-1}$ whereas the corresponding expression in the standard model is given by $H_s = \frac{1}{2t}$. One can therefore, find that $H \lesseqgtr H_s$ according as $R \lesseqgtr \frac{4}{3}R_0$. The parameter R_0 may therefore be chosen suitably, consistent with $\rho_{r0}G_0R_0^2 = \frac{3}{32\pi}$ which is the only constraint involving ρ_{r0}, G_0 and R_0, to have sufficient H through the epoch of light element production in order to avoid any danger of inappropriate element abundances.

Assuming that the radiation is associated with its temperature T by

$$\rho_r = \frac{\pi^2}{15}T^4 \text{ (in suitable units)}, \tag{3.6}$$

we find that T is given by

$$T = \left[\frac{15\rho_{r0}}{\pi^2}\right]^{\frac{1}{2}}\left[\frac{t^2}{4R_0^2} + 1\right]^{-1} \tag{3.7}$$

Thus, the maximum temperature $T_{max} = T_0 = (15\rho_{r0}/\pi^2)^{1/4}$. As the universe is spatially finite in this case ($k = 1$), it is possible to determine the time $t = t_{cau}$ when the whole universe becomes causally connected. This is given by [2,8]

$$\int_0^{t_{cau}} \frac{dt}{R(t)} = \int_0^1 \frac{dr}{\sqrt{(1-r^2)}} = \frac{\pi}{2}. \tag{3.8}$$

By the use of (3.5), this gives

$$t_{cau} = 2R_0. \tag{3.9}$$

It may be noted that the model in this case, also describes a phase of contraction before the moment of rest, as is clear from eq. (3.5), with an isotropic and homogeneous distribution of matter.

(b) A linearly expanding scenario :

For $x = \frac{1}{2}$, eqs. (3.2) and (3.3) respectively, reduce to

$$\dot{R} = \left[\frac{16\pi AE_r}{3} - k\right]^{\frac{1}{2}} = \text{constant (say } (\dot{R})_0), \tag{3.10}$$

and
$$\ddot{R} = 0. \tag{3.11}$$

Eq. (3.10) demands $16\pi AE_r > 3k$ and yields the solution

$$R = (\dot{R})_0 t, \tag{3.12}$$

choosing $R_0 = 0$ and hence $G_0 = 0, \rho_{r0} = \rho_{v0} = \infty$.

The horizon distance $d_H(t)$ at time t is the proper distance travelled by light emitted at $t = 0$:

$$d_H(t) = R(t) \lim_{t_0 \to 0} \int \frac{dt'}{R(t')} \tag{3.13}$$

This for the solution (3.12), obtains

$$d_H(t) = t \lim [\ln(t/t_0)] = \infty, \tag{3.14}$$

showing that the model has no horizon, irrespective of the value of k. We here note that the present model in the early phase of evolution reduces to the one obtained by us [9] by an altogether different parametrization.

(c) Closed, open or flat big bang scenario :

For $x = \frac{1}{4}$, the evolution is described by

$$\dot{R}^2 = \left(\frac{32\pi AE_r}{9}\right)\frac{1}{R} - k, \tag{3.15}$$

which follows from eq. (3.2). Eq. (3.15) yields the solutions

$$t = -\sqrt{(BR - R^2)} + \frac{B}{2}\sin^{-1}\left(\frac{2R}{B} - 1\right) - \frac{3\pi B}{4}, \text{ when } k = 1, \tag{3.16}$$

$$= \sqrt{(BR + R^2)} - \frac{B}{2}\cosh^{-1}\left(\frac{2R}{B} + 1\right), \text{ when } k = -1, \tag{3.17}$$

and

$$R = \left(\frac{3\sqrt{B}}{2}t\right)^{2/3}, \text{ when } k = 0. \tag{3.18}$$

Here

$$B = \frac{32\pi AE_r}{9} \text{ and } R_0 = 0.$$

(d) Inflationary and big bang scenarios with k = 0 :

For $k = 0$, eq. (3.2) yields the solution

$$R = \left[\sqrt{\{32\pi AE_r(1-x)/3\}}t + R_0^{2(1-x)}\right]^{1/2(1-x)} \tag{3.19}$$

The evolution is inflationary for $x > \frac{1}{2}$, as we have mentioned earlier, and starts from R_0 with $H_0 = \sqrt{\{8\pi A E_r / 3(1-x)\}} R_0^{2(x-1)}$, $G_0 = AR_0^{4x}$ and $\rho_{v_0} = \frac{x}{(1-x)} \rho_{r_0}$. However, the choice $H_0 = 0$ leads to $R_0 = 0$ and hence $G_0 = 0$. For $x < \frac{1}{2}$, the expansion turns deflationary ($\ddot{R} < 0$) and the evolution consequently starts from a big bang at $R_0 = 0$ as mentioned earlier.

It is interesting to note that for $x < \frac{1}{2}$, the variation of the scale factor R given by (3.19), is the same as in the model of Freese *et al* [3]. However, the essential difference is there in the variation of ρ_r in the two models. For example, taking $x = \frac{1}{4}$ in the model of Freese *et al*, amounts to $\rho_r \sim R^{-3}$ whereas in the present model, $\rho_r \sim R^{-4}$ independently of the value of x.

4. The present universe

We note from eq. (2.11) that ρ_r as well as $\dot{\rho}_r$ can be neglected compared to ρ_m for a sufficiently large R though these may otherwise be small non-zero quantities. This leads to

$$\rho R^3 \simeq \text{constant, in the M.D. era.} \tag{4.1}$$

Also eq. (2.8) by the use of (1.1), (2.6) and (2.7), in this context, of further neglecting ρ_r and $\dot{\rho}_r$ and in comparision to ρ_m and $\dot{\rho}_m$, obtains

$$G \rho R^3 \simeq \text{constant}, \tag{4.2}$$

which with (4.1) leads to

$$G \simeq \text{constant} \; (\equiv G_{\text{present}}), \tag{4.3}$$

which is in agreement with (2.15). ρ_v (as well as Λ) is then also a small (negligible compared to ρ_m) constant quantity as eq. (2.12) indicates. Thus the model in the M.D. era, approaches the standard model which is considered to give the best representation of the present universe.

5. Concluding remarks

With a variable G and a dynamically decaying vacuum energy, which decays as the radiation content of the universe, we find that in the early R.D. era, the decaying vacuum lifts gradually the gravitational coupling G from its minimum (non-zero when the model starts form a non-singular origin and zero when it starts from a singularity), promotes it as R^{4x}, $0 < x < 1$ and makes it approach a maximum present value G_p as the universe turns matter dominated. We also find that for $\rho_v > \rho_r$, the model may have a non-singular origin with maximum temperature, radiation and vacuum energy densities and the expansion in this case would be endowed with some general type of inflation in the early universe. For $\rho_v = \rho_r$ (with $k = 0, \pm 1$) and $\rho_v = 3\rho_r$ (with $k = 1$), the model is free from the horizon problem. The present models are interesting as they ultimately evolve to the standard big bang models in the present phase though their early scenarios are altogether different.

References

[1] A D Sakharov *Sov Phys. Dokl* **12** 1040 (1968); P J E Peebles and B Ratra *Astrophys. J* **325** L17 (1988)

[2] M Ozer and M O Taha *Phys Lett.* **B171** 363 (1986), *Nucl. Phys* **B287** 776 (1987)

[3] K Freese, F C Adams, J A Frieman and E Mottola *Nucl Phys.* **B287** 797 (1987)

[4] M Gasperini *Phys Lett* **B194** 347 (1987); *Class Quant. Gravit* **5** 521 (1988); W Chen and Y S Wu *Phys Rev* **D41** 695 (1990), M S Berman *Phys. Rev* **D43** 1075 (1991), J C Carvalho, J A S Lima and I Waga *Phys. Rev* **D46** 2404 (1992), E W Kolb and M S Turner *The Early Universe* (New York Addition-Wesley) (1990), Abdussattar and R G Vishwakarma *Indian J Phys.* **69B** 431 (1995); *Indian J Phys* **70B** 75 (1996), *Pramana J Phys* **47** 41 (1997)

[5] P S Wesson *Cosmology and Geophysics* (Oxford · Oxford University Press) (1978); *Gravity, Particles and Astrophysics* (Dordrecht : Reidel) (1990)

[6] D Kalligas, P Wesson and C W F Everitt *Gen Relativ Gravit* **24** 351 (1992), A M M Abdel-Rahman *Gen. Relativ Gravit* **22** 655 (1990), M S Berman *Gen Relativ Gravit.* **23** 465 (1991), A Beesham *Int J Theor Phys* **25** 1295 (1986); Abdussattar and R G Vishwakarma *Indian J Phys* **70B** 321 (1996), *Class Quant Gravit* **14** 945 (1997), *Aust J Phys.* **50** (5) (to appear) (1997)

[7] P G Bergmann *Int J Theor Phys* **1** 25 (1968), R V Wagoner *Phys Rev* **D1** 3209 (1970), A D Linde *JETP Lett* **19** 183 (1974), M Der Sarkissian *Nuovo Cim* **B88** 29 (1985), A M M Abdel Rahman *Nuovo Cim* **B102** 225 (1988), D Wilkins *Am J. Phys* **54** 726 (1986)

[8] S Weinberg *Gravitation and Cosmology* (New York Wiley) (1971)

[9] Abdussattar and R G Vishwakarma *Class Quant Gravit* **14** 945 (1997), *Aust J Phys* **50** (5) (to appear) (1997)